

THE ROLE OF MATHEMATICS IN PHYSICS

by

C. S. SHARMA

Department of Mathematics, Birkbeck College, University of London.

Michael has asked me
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1. INTRODUCTION

My main contention is that it is imperative that physicists should modernize their knowledge of mathematics. In order to explain why this is so, it is necessary to examine what mathematics is and what its role in physics is. What I have to say is said by a practising mathematical physicist to other mathematical physicists and if what I have to say can provide anything of interest to philosophers, then I shall, of course, be delighted. However, I must confess that I have no formal training in either philosophy or methodology.

Few mathematicians or physicists are ever told what mathematics is - it is assumed to be what they learn in their mathematics courses. This is most unsatisfactory, particularly because students of modern mathematics are repeatedly told that in mathematics existence is synonymous with freedom from contradictions and in mathematics only those things or concepts exist which can be defined in a contradiction free way. No attempt is made to define mathematics, presumably because of the fear that it cannot be done in a contradiction free way and if a student establishes a contradiction in the definition given to him in relation to the mathematics course he is taking, then it will have to be conceded that he is being trained in a non-existent discipline. I am talking here about the usual text-books and the usual courses. This is not to say that no one has ever attempted to define mathematics, on the contrary as is well-known there are tomes written on the definition and philosophy of

mathematics.

Kline [1] has written: "... mathematics is a body of knowledge. But it contains no truths." Some philosophers such as Robinson [2] and Cohen [3] regard mathematics as a mere manipulation of symbols according to a set of rules. Yet another school of philosophers [4-5] regard mathematics as nothing else but certain intuitive constructions with the help of symbols. Logicians such as Frege [6] and Russell [7] regard mathematics as that self-contained branch of knowledge which contains certain truths, the arguments which establish the truths, the constructions which underly these arguments and the formal manipulation of symbols that express these arguments and truths. This definition is criticized on the basis that it does not assert what the truths are about and the shortcoming is remedied by defining mathematics as the study of logical truths about abstract structures [8]. There may be differences of opinion about what mathematics is, but mathematicians seem to agree that mathematics, by which I mean pure mathematics, has nothing to do with the reality of the world as perceived by our senses and that it is independent of other sciences which study such reality. On the contrary the traditional view in this country from the time of Newton has been that science is the study of the nature of reality and different branches of science merely study different aspects of the same reality: mathematics being the branch which is concerned with the quantitative aspects of reality exemplified in its simplest form by various kinds of measurements. It was probably the bigamous wedding of mathematics to logic which led later to its divorce from other sciences. Mathematicians began to gloat over the claim that the art or the science or whatever mathematics is has nothing to do with reality and has no use whatsoever. They altered their

language by adopting numerous symbols from logic. The change happened over a period of time in slow stages and there are still a few university departments which teach pure mathematics of the kind which is neither formal nor divorced from other sciences, but the new formal pure mathematics is so sleek and so elegant that it attracted all the clever young men and the pure mathematics of the old kind is beginning to disappear from the curricula. While the changeover from the old to the new has been taking place, it has been a source of considerable strain and conflict between applied mathematicians and pure mathematicians resulting in an unfortunate parting of ways exemplified in numerous cases by the placing of applied mathematicians in a separate department with the theoretical physicists. While physicists and engineers seemed to show a distinct preference for the mathematics of the older kind, practitioners of social and economic sciences which have come into their own fairly recently found that the new mathematics was quite useful in making both their qualitative and quantitative arguments more obscure for laymen and the term modern applied mathematics began to be used to describe various recently developed techniques, which compared to what physicists and engineers do, can only be described as relatively trivial applications of mathematics. Much more recently further changes have begun to take place. Divorced from reality, study of pure mathematics became arid and obscure and it became increasingly difficult to do research in pure mathematics. In days of affluence the lack of a significant number of papers from a department of pure mathematics can be a matter of some misplaced pride among its members, but in the present climate of economic recession this has become a matter of grave concern. Both economic necessity and a genuine feeling of a loss of a great fountain of inspiration have led an increasing number of pure mathematicians into seeking out

theoretical physicists in an attempt to reestablish once again the intimate alliance which used to exist between physics and pure mathematics. One can mention the recent activities of Atiyah and Kaplansky as examples of this trend: both were practising pure mathematicians of the greatest distinction and in recent years both have been finding the greatest enthusiasm in collaborating with theoretical physicists.

In my opinion, the divorce between mathematics and physics has been even more damaging for the physicist. It was particularly unfortunate that this should happen after both quantum theory and relativity had become established after delivering mortal blows to the deterministic view of science and to the absolute. Quantum theory is based on the structure of operators on a Hilbert space which is a very abstract structure indeed and relativity uses a geometry which defies intuition. Due to a great misfortune, in my opinion, it has been possible to do quantum theory without using Hilbert space theory and physicists have not been paying much attention to general relativity: for doing special relativity, which physicists found impossible to ignore, all one needs is the Lorentz transformation and a reluctant acceptance of certain peculiar views about simultaneity. However, the time is fast approaching when the limitations of quantum theory will make it imperative to find a better theory to which quantum theory in its present form is an approximation and the study of quantum fields has made it clear that in all probability the new theory when it comes will have to use as its basis a structure which is much more sophisticated than that of a Hilbert space. Also space travel has rekindled interest in astrophysics and it has become clear that one cannot really understand his universe without learning something about general relativity. Relativity was the first physical theory

which used a supposedly unreal geometry and relativists have found it necessary to grapple with topology and manifold theory to improve their understanding of their world. The majority of relativists now-a-days use modern mathematics - not merely because it is in fashion^{but} because it is necessary so to do. On the question of fashions, my late supervisor at Oxford used to think that the use of modern mathematics in physics was a fashion which will pass. There are any number of theoretical physicists and chemists who would strongly disagree with me when I say that my late supervisor was wrong on this point. However, I do not think that there is anyone who would not find the following quotation as an amusing piece of absolute nonsense [9]: "It is hardly possible to ignore the present-day fashion of using vectors in Dynamics; and, to meet the needs of students who wish to know how to obtain the fundamental equations of dynamics by vectorial methods, I have introduced the subject and taken it thus far in an appendix. I doubt whether it is a fashion which has come to stay, and I am unwilling to require all readers of this book to use vectors. I tried to fit them in, in several places in the body of the book, but eventually decided that an appendix would be the best place for them." Vectors came to stay in the study of dynamics not because a fashion became a habit but because they not only made calculations simpler but also provided genuine physical insight into dynamical situations.

We need a philosophy of mathematics and its application in physics which will remove the conflicts which have damaged, by slowing down, the progress of both mathematics and physics. I hope, I have an outline of such a philosophy. However, because of lack of training in methodology and philosophy, I cannot isolate and examine the shortcomings of my views. I shall first describe

what in my view mathematics is and then I shall deal with the role of mathematics in physics.

2. What is mathematics

I had studied a great deal of mathematics before meeting a definition of mathematics. I should have gone to a book on mathematical logic or on foundations of mathematics, but a mathematical physicist has to learn both physics and mathematics and if one is as lazy as I am, one does not get the time to study everything one would like to or one ought to study. I somewhere caught the phrase that mathematics is a study of abstract structures which appealed to me and which I now know is based on the so-called platonist view on the philosophy of mathematics. Though I use the same phrase, my views differ on a number of important points ~~with~~ those held by the platonists. Study of abstract structures: it is a nice phrase indeed, but what does it mean? What is an abstract structure? What is abstract? What is a structure? I think those who claim that mathematics has nothing to do with reality use the word abstract to mean abstruse. Most applied mathematicians who have not been alienated by modern mathematics and many pure mathematicians who find inspiration for their art in real concrete structures would agree with me that the word abstract means "essence". We study a real concrete physical structure and abstract the essential aspects of the structure to get an abstract structure. This still does not mean much because we have not assigned a definite meaning to the word structure. We all vaguely know what the word structure means and if one consults a dictionary one finds other words or combination of words to express what is meant by structure, for example, the Concise Oxford Dictionary gives the following meanings of the word structure: 1. manner in which a building or organism or other complete whole is constructed, 2. supporting

framework, 3. whole of the essential parts of something, 4. make and 5. construction. The word structure, of course, means each and all of these things, but any particular structure arises because of the peculiar relationships which exist between its constituent parts and I would define the word structure to mean the relationships between the constituent parts of a whole. A structure is then a collection of relationships between objects, the objects capable of sustaining a peculiar relationship can be collected together to form a naive set and the relationship provides the set with a structure. In order to illustrate the point I consider a few examples. It does not require much abstracting from the primitive art of counting to get the structure of natural numbers, once the relationship involved in counting is recognized as a bijection between the collection of objects being counted and the set of first so many natural numbers, logical analysis with a little help from a very plausible axiom of choice easily leads one to see the existence of transfinite cardinal numbers. By abstracting ^{from} the art of valuing assets and liabilities in everyday commerce and industry, we get the set of rational numbers and the binary relations of addition and multiplication. Out of these we take just the positive and negative integers and study merely the relationship involved in the addition of integers and abstract the essential aspects of it to get the definitive axioms of a commutative group. We study the symmetries of a crystal under rotation and find that in essence the structure involved has much in common with the addition of integers, though commutativity fails to hold; we discard the requirement of commutativity from the axioms of a commutative group and get the axioms of a more general structure called simply a group. By logical analysis one finds a whole series of properties which go with a

particular abstract structure and whenever a physical structure when studied carefully is found to contain the essence of the abstract structure, it is bound to have all the properties which have been established for the abstract structure. By abstracting, we simplify the task of studying the properties of a large variety of apparently unrelated physical structures, if the basic relationship in a variety of physical structures satisfies the definitive axioms of an abstract structure, then they must share the same basic properties which may manifest themselves physically in different ways. When the definitive axioms of an abstract structure arising out of the study of an actual physical structure are altered either for the purpose of gaining greater insight into the situation or (for just) fun, a new abstract structure is created which apparently has no counterpart in reality: such structures I call abstruse structures. It is a remarkable fact that most abstruse structures created in this way are later found to share the structure of certain aspects of reality which had not been known or understood before and when this happens to a particular abstruse structure, it loses its abstruseness. In this way abstract structures and reality are closely related. I hope it is clear by now what I mean by the term abstract structure and mathematics is the study of abstract and abstruse structures. From all this it should be clear that the study of mathematics is going to be very profitable in the study of the structure of physical reality. I next consider the role of mathematics in physics.

3. The role of mathematics in physics

Physics is the study of the structure of physical reality. I agree with the positivists in asserting that anything we know about physical reality with certainty is based on experimental observations and that we cannot have any a priori knowledge about physical reality.

The experimental physicist uses not only his senses but all kinds of instruments to observe the particular aspect of physical reality he is interested in - he observes and measures and then he looks for regularities and patterns in the collection of his observations and he tries to fit them to accord with laws which he postulates to explain these patterns. In classical physics, mathematics has been regarded as the language in which the quantitative aspects of the observations and the laws postulated to correlate them are expressed. It was a fundamental belief among physicists that both the absolute and the exact were, at least in principle, attainable in the study of physical reality and that some of the laws which had been discovered were in fact both exact and absolute. It was believed that the laws of nature were simple both in their qualitative and quantitative aspects and could be grasped and understood intuitively. During the last century physicists were forced to revise drastically their views on the laws of nature. The discovery of relativity destroyed the absolute and that of quantum theory the exact. The only laws of nature which are known to be either exact or absolute or both can ^{be} easily shown to be tautologies arising from particular definitions of technical terms and are not really laws of nature. To illustrate this point I shall state just one law: if two bodies at different temperatures are connected by a thermal conductor, then heat flows from the one which is at higher temperature to the one which is at lower temperature. I claim that this is absolutely true because of the meaning we ascribe to the word temperature and has nothing to do with the structure of physical reality. Not only we cannot attain the exact and the absolute, but both relativity and quantum theory use mathematical models about which we have, at any rate, very limited intuition and which are based on abstract structures which

we know not as a result of abstraction from their respective physical counterparts but ^{they} arose in a way which could qualify them for the title of abstruse structures in the sense I have defined the term abstruse earlier. In order to understand those aspects of physical reality which are described by quantum theory and relativity, it is now necessary to understand the structures of operators on a Hilbert space and of tensor fields on differentiable manifolds and much of the information comes from a formal logical analysis which is intuitively almost impossible to grasp. It is, therefore, necessary to look at the role of mathematics in physics a little more carefully.

When we make observations, we gain information about a particular aspect of physical reality. In order to gain a more complete knowledge we have to combine together information obtained from a variety of experiments: when we combine all the information we have we obtain a description of the subject matter of our study. Any description is not the real thing: it is only a model. It is a model because it is both idealized and imperfect. It is idealized in the sense that it ignores those aspects of the actual structure which are not contained in the description and it is imperfect in the sense that it is describing a knowledge which is both incomplete and inexact. The description is in fact the result of an abstraction of the kind I defined while defining an abstract structure: the description identifies the parts which constitute the physical system under study and describes the interrelations which exist between them. When we abstract once again by looking carefully at the essential aspects of the relationships which constitute the structure and abandon the particular objects which are so related we get an abstract mathematical structure which models the physical system. If this view is correct then the role of mathematics in

physics is primarily to provide mathematical models for describing physical reality. The aim of the mathematical physicist is to find that abstract structure which has the same essential properties as the particular physical structure which he is studying. It is manifestly obvious that qualitative and quantitative aspects of the structure are intimately related and in the mathematical model one has both the qualitative and the quantitative aspects of the structure. What I have just said is valid for the simplest observations. Models play a much more intimate role in modern physics: the actual observations on dials, graphs, photographic plates and digital counters are numbers and curves which by themselves convey little direct information about the actual observations they are supposed to make. It is only when they are interpreted on the basis of quite a sophisticated model that they begin to be meaningful observations. The belief that the underlying models provide meaningful descriptions is based on deductions which may not be logically rigorous but are plausible. A good physicist is always aware of the model in the background; a bad one confuses the model for the real thing.

Once a mathematical model has been discovered, it will have with the actual physical structure all those things in common which led to its discovery, but it will have aspects which exist because they are a logical necessity arising from the definition of the abstract structure but which are not known to correspond to any aspect of physical reality. If the model is good, by which I mean that its defining relationships are good descriptions of actual relationships in the physical structure, then the logical consequences of the relationships are likely to have counterparts in the physical structure as well. An inspired guess as to how the abstract model and the physical thing are related in areas

which have not been experimentally studied, gives rise to the so-called experimentally verifiable predictions. Many such predictions have been successfully made in the past, but the predictions are essentially speculative in the sense that we cannot be absolutely certain that they will be found to be true until they are eventually found to be so. When predictions fail and unfortunately this happens more frequently than they succeed, then unless we have made a mistake, we come to the area of the model where it is beginning to be a bad description of the actual thing. In order to arrive at a better model, it is necessary to understand both the structures: the real as well as the abstract with all their pathologies and in order to do this we are likely to need all the help that modern pure mathematics can provide. However, it is true that the mathematical model of quantum theory was discovered not as a result of a mathematical and logical analysis of the shortcomings of the Newtonian model, but by a kind of reasoning which required all the speculative genius which the founding fathers had. The majority of physicists have found it possible to practise quantum theory without learning much about the structure of operators on a Hilbert space; this makes many physicists inclined to think that modern pure mathematics has no relevance for them. Even though it is possible to do a kind of quantum theory without knowing the intricacies of the spectral theory of operators, it is now impossible to do much without the help of group theory and its representations and those who have discovered the simplicity of the abstract version will need little persuasion that a knowledge of abstract pure mathematics makes the task of the mathematical physicist easier. The symbolism of modern mathematics much of which came from logic is repugnant to many physicists, but every physicist agrees about the necessity of

thinking and talking logically and of stating one's problems as precisely as possible - both tasks are made much easier by using the language and symbols which have evolved as a result of interaction between mathematics and logic and have enriched the treasury of tools available to the physicist. I do not think that the use of symbols in most modern mathematical texts is excessive.

I should perhaps concede that major advances in theoretical physics are often based on computational algorithms based on concepts or techniques which contradict the rules of modern mathematics. I shall illustrate this point by two examples: 1. The computational algorithm in quantum theory based on the Dirac δ -function has played a very important part in the development and the interpretation of quantum theory and 2. The computational technique in quantum theory based on the ad-hoc rule that a wave function and its complex conjugate can be varied independently of each other is a very useful one in solving a variety of problems by using the calculus of variations. Neither of them make sense in terms of modern mathematics. There is no harm in continuing to use such algorithms for doing one's calculations, but being inconsistent they do not provide an explanation or a deep understanding of the underlying physical reality and it is a task for the mathematician to remove the inconsistencies by developing appropriate mathematical structures which will remedy the situation. Dirac's δ -function led to the discovery of the theory of distributions and the rule that a wave function and its complex conjugate could be varied independently of each other led to the discovery of a new calculus on complex Banach spaces. If any description however inconsistent reproduces some aspect of reality, we have to learn from it. In order to gain a deeper understanding and to remove the contradictions, one will have to analyze the situation at depth and it is my contention that one is

likely to need all that mathematics and logic have to offer for so doing.

It is a pity that physicists did not take up the study of general relativity sooner than they did. Even in its older forms general relativity has depended on manipulation of symbols which were almost as subtle as anything in symbolic logic and general relativity has always used models which are intuitively rather difficult to grasp. More recently in order to understand the nature of black holes it has been found essential to study the nature of singularities and this at the present time can be done only by borrowing techniques and concepts from modern pure mathematics. With the rise of interest in general relativity physicists are less inclined to argue against my main contention than they were only a few years ago.

I once again come back to the strange fact that it is possible to do quantum theory without knowing any modern pure mathematics only to point out that this is possible only at a price. I shall illustrate my point by three examples:

1. In 1968 Herzenberg and Lau [10] wrote a paper which in its original version had a very long appendix. The result in the appendix was attributed to a third scientist who is extremely distinguished and a fellow of the Royal Society of long standing: the result which was said to be important was proved by a series of tedious and tortuous arguments and calculations. In fact the result was completely trivial and could be proved in about four lines by using first year linear algebra. The referee pointed this out to the authors who were both embarrassed and grateful and in the published version the short linear algebra proof was given.

2. Some years ago a paper was written by a brilliant Cambridge trained mathematical physicist. It was a long and complicated

paper and all the arguments given were correct. It represented a considerable piece of work. However, the final achievement of the work was a result which followed in one line from the Cauchy-Schwarz inequality. The referee pointed this out to the author and the author withdrew the paper.

3. Cohen, Feldmann and McEachran published a paper [11] in 1972 in which they claimed to find improved lower bounds to certain atomic energy levels, but the calculation of the bounds required a prior knowledge of the energies. The referee pointed out that if an energy is known (experimentally or otherwise), it is itself its own best lower and upper bound and asked why it was necessary to do all the complicated calculations reported in the paper to produce a substantially inferior lower bound. The editors of the journal accepted the paper because even though it was impossible to counter the referee's objection, the work of Cohen et al was supposed to throw light on some mystical aspects of certain mythical objects called orbitals. The expression "a record in nonsense" has been used to describe this work, but every year at least a thousand papers are published which are even more nonsensical than the work of Cohen et al.

I have often been asked the question: "If the mathematical methods I know are enough for my needs, is it necessary for me to learn new and strange methods?" There are many research workers who spend their lives working out numbers using a well-defined algorithm based on some theory which is often, though not always, ill-founded. If such workers are happy and satisfied with what they are doing it is certainly not necessary for them to learn new methods but they are quite likely to be replaced by a few floppy discs. In practical use of mathematics in the laboratory or in

the industry, the mathematics used in most cases is actually an algorithm for turning a set of data to another set of data and in the coming years such tasks will be increasingly done by magnetic tapes, floppy discs ^{and} plastic cards with the help of silicon chips.

On the frontiers of knowledge, where mathematical models begin to break down, particularly in view of the extremely sophisticated models being used by physicists, it is necessary that the problems involved should be defined with as much precision and clarity as possible and I believe that the language and symbolism of modern mathematics is the most appropriate and the best vehicle available for carrying such a description of the problems.

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